Some Results on incidence coloring, star arboricity and domination number

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Abstract

Two inequalities bridging the three isolated graph invariants, incidence chromatic number, star arboricity and domination number, were established. Consequently, we deduced an upper bound and a lower bound of the incidence chromatic number for all graphs. Using these bounds, we further reduced the upper bound of the incidence chromatic number of planar graphs and showed that cubic graphs with orders not divisible by four are not 4-incidence colorable. The incidence chromatic numbers of Cartesian product, join and union of graphs were also determined.

1 Introduction

An incidence coloring separates the whole graph into disjoint independent incidence sets. Since incidence coloring was introduced by Brualdi and Massey [4], most of the researches were concentrated on determining the minimum number of independent incidence sets, also known as the incidence chromatic number, which can cover the graph. The upper bound of the incidence chromatic number of planar graphs [9], cubic graphs [10] and a lot of other classes of graphs were determined [9, 8, 12, 14, 17]. However, for general graphs, the best possible upper bound is an asymptotic one [6]. Therefore, to find an alternative upper bound and lower bound of the incidence chromatic number for all graphs is the main objective of this paper.

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In Section 2, we will establish a global upper bound for the incidence chromatic number in terms of chromatic index and star arboricity. This result reduces the upper bound of the incidence chromatic number of the planar graphs. Also, a global lower bound which involves the domination number will be introduced in Section 3. Finally, the incidence chromatic number of graphs constructed from smaller graphs will be determined in Section 4.

All graphs in this paper are connected. Let $V(G)$ and $E(G)$ (or V and E) be the vertex-set and edge-set of a graph G, respectively. Let the set of all neighbors of a vertex u be $N_G(u)$(or simply $N(u)$). Moreover, the degree $d_G(u)$(or simply $d(u)$) of u is equal to $|N_G(u)|$ and the maximum degree of G is denoted by $\Delta(G)$ (or simply Δ). All notations not defined in this paper can be found in the books [3] and [18].

Let $D(G)$ be a digraph induced from G by splitting each edge $e(u, v) \in E(G)$ into two opposite arcs uv and vu. According to [12], incidence coloring of G is equivalent to the coloring of arcs of $D(G)$. Two distinct arcs uv and xy are *adjacent* provided one of the following holds:

(1) $u = x$;
(2) $v = x$ or $y = u$.

Let $A(G)$ be the set of all arcs of $D(G)$. An *incidence coloring* of G is a mapping $\sigma : A(G) \to C$, where C is a *color-set*, such that adjacent arcs of $D(G)$ are assigned distinct colors. The *incidence chromatic number*, denoted by χ_i, is the minimum cardinality of C for which $\sigma : A(G) \to C$ is an *incidence coloring*. An *independent set* of arcs is a subset of $A(G)$ which consists of non-adjacent arcs.

2 Incidence chromatic number and Star arboricity

A *star forest* is a forest whose connected components are stars. The *star arboricity* of a graph G (introduced by Akiyama and Kano [1]), denoted by $st(G)$, is the smallest number of star forests whose union covers all edges of G.

We now establish a connection among the chromatic index, the star arborcity and the incidence chromatic number of a graph. This relation, together with the results by Hakimi et al. [7], provided a new upper bound of the incidence chromatic number of planar graphs, k-degenerate graphs and bipartite graphs.

Theorem 2.1 *Let G be a graph. Then $\chi_i(G) \le \chi'(G) + st(G)$, where $\chi'(G)$ is the chromatic index of G.*

Proof: We color all the arcs going into the center of a star by the same color. Thus, half of the arcs of a star forest can be colored by one color. Since $st(G)$ is the smallest number of star forests whose union covers all edges of G, half of the arcs of G can be colored by $st(G)$ colors. The uncolored arcs now form a digraph which is an orientation of G. We color these arcs according to the edge coloring of G and this is a proper incidence coloring because edge coloring is more restrictive. Hence $\chi'(G) + st(G)$ colors are sufficient to color all the incidences of G. $\square$

We now obtain the following new upper bounds of the incidence chromatic numbers of planar graphs, a class of k-degenerate graphs and a class of bipartite graphs.

Corollary 2.2 *Let G be a planar graph. Then $\chi_i(G) \le \Delta + 5$ for $\Delta \ne 6$ and $\chi_i(G) \le 12$ for $\Delta = 6$.*

Proof: The bound is true for $\Delta \le 5$, since Brualdi and Massey [4] proved that $\chi_i(G) \le 2\Delta$. Let G be a planar graph with $\Delta \ge 7$, we have $\chi'(G) = \Delta$ [11, 16]. Also, Hakimi et al. [7] proved that $st(G) \le 5$. Therefore, $\chi_i(G) \le \Delta + 5$ by Theorem 2.1. $\square$

While we reduce the upper bound of the incidence chromatic number of planar graphs from $\Delta + 7$ [9] to $\Delta + 5$, Hosseini Dolama and Sopena [8] reduced the bound to $\Delta + 2$ under the additional assumptions that $\Delta \ge 5$ and girth $g \ge 6$.

A *k-degenerate graph* G is a graph with vertices ordered $v_1, v_2, \ldots, v_n$ such that each v_i has degree at most k in the graph $G[v_1, v_2, \ldots, v_i]$. A *restricted k-degenerate graph* is a k-degenerate graph with the graph induced by $N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$ is complete for every i. It has been proved by Hosseini Dolama et al.[9] that $\chi_i(G) \le \Delta + 2k - 1$, where G is a k-degenerate graph. We lowered the bound for restricted k-degenerate graph as follow.

Corollary 2.3 *Let G be a restricted k-degenerate graph. Then $\chi_i(G) \le \Delta + k + 2$.*

Proof: By Vizing's theorem, we have $\chi'(G) \le \Delta + 1$. Also, the star arboricity of a restricted k-degenerate graph G is less than or equal to $k + 1$ [7]. Hence we have $\chi_i(G) \le \Delta + k + 2$ by Theorem 2.1.

Corollary 2.4 *Let B be a bipartite graph with at most one cycle. Then $\chi_i(B) \leq \Delta + 2$.*

Proof: Hakimi et al. [7] proved that $st(B) \leq 2$ where B is a bipartite graph with at most one cycle. Also, it is well known that $\chi'(B) = \Delta$. These results together with Theorem 2.1 proved the corollary. $\square$

3 Incidence chromatic number and Domination number

A *dominating set* $S \subseteq V(G)$ of a graph G is a set where every vertex not in S has a neighbor in S. The *domination number*, denoted by $\gamma(G)$, is the minimum cardinality of a domination set in G.

A *maximal star forest* is a star forest with maximum number of edges. Let $G = (V, E)$ be a graph, the number of edges of a maximal star forest of G is equal to $|V| - \gamma(G)$ [5]. We now use the domination number to form a lower bound of the incidence chromatic number of a graph. The following proposition reformulates the ideas in [2] and [10].

Proposition 3.1 *Let $G = (V, E)$ be a graph. Then $\chi_i(G) \geq \dfrac{2|E|}{|V| - \gamma(G)}$.*

Proof: Each edge of G is divided into two arcs in opposite directions. The total number of arcs of $D(G)$ is therefore equal to $2|E|$. According to the definition of the adjacency of arcs, an independent set of arcs is a star forest. Thus, a maximal independent set of arcs is a maximal star forest. As a result, the number of color class required is at least $\dfrac{2|E|}{|V| - \gamma(G)}$. $\square$

Corollary 3.2 *Let $G = (V, E)$ be an r-regular graph. Then $\chi_i(G) \geq \dfrac{r}{1 - \frac{\gamma(G)}{|V|}}$.*

Proof: By Handshaking lemma, we have $2|E| = \sum_{v \in V} d(v) = r|V|$, the result follows from Proposition 3.1. $\square$

Corollary 3.2 provides an alternative method to show that a cycle C_n, where n is not divisible by 3, is not 3-incidence colorable. As C_n is a 2-regular graph with $\gamma(C_n) > \dfrac{|V(C_n)|}{3}$, we have

$$\chi_i(C_n) \geq \frac{2}{1 - \frac{g(G)}{|V|}} > \frac{2}{1 - \frac{1}{3}} = 3.$$

Corollary 3.3 *Let $G = (V, E)$ be an r-regular graph. Two necessary conditions for $\chi_i(G) = r + 1$ (also for $\chi(G^2) = r + 1$ [14]) are:*

1. *The number of vertices of G is divisible by $r + 1$.*

2. *If r is odd, then the chromatic index of G is equal to r.*

Proof: We prove 1 only, 2 was proved in [13]. By Corollary 3.2, if G is an r-regular graph and $\chi_i(G) = r + 1$, then $r + 1 = \chi_i(G) \geq \dfrac{r|V|}{|V| - \gamma(G)} \Rightarrow \dfrac{|V|}{r+1} \geq \gamma(G)$. Since the global lower bound of domination number is $\left\lceil \dfrac{|V|}{\Delta + 1} \right\rceil$, we conclude that the number of vertices of G must be divisible by $r + 1$. □

4 Graphs Constructed from Smaller Graphs

In this section, we determine the upper bound of the incidence chromatic number of union of graphs, Cartesian product of graphs and join of graphs, respectively. Also, these bounds can be attained by some classes of graphs [15]. Let the set of colors assigned to the arcs going into u be $C_G^+(u)$. Similarly, $C_G^-(u)$ represents the set of colors assigned to the arcs going out from u.

We start by proving the following theorem about union of graphs.

Theorem 4.1 *For all graphs G_1 and G_2, we have $\chi_i(G_1 \cup G_2) \leq \chi_i(G_1) + \chi_i(G_2)$.*

Proof: If some edge $e \in E(G_1) \cap E(G_2)$, then we delete it from either one of the edge set. This process will not affect $I(G_1 \cup G_2)$, hence, we assume $E(G_1) \cap E(G_2) = \varnothing$. Let σ be a $\chi_i(G_1)$-incidence coloring of G_1 and l be a $\chi_i(G_2)$-incidence coloring of G_2 using different color set. Then ϕ is a proper $(\chi_i(G_1) + \chi_i(G_2))$-incidence coloring of $G_1 \cup G_2$ with $\phi(uv) = \sigma(uv)$ when $e(uv) \in E(G_1)$ and $\phi(uv) = l(uv)$ when $e(uv) \in E(G_2)$. □

The following example revealed that the upper bound given in Theorem 4.1 is sharp.

Example 4.1 Let n be an even integer and not divisible by 3. Let G_1 be a graph with $V(G_1) = \{u_1, u_2, \ldots, u_n\}$ and $E(G_1) = \{u_{2i-1}u_{2i} | 1 \leq i \leq \frac{n}{2}\}$. Furthermore, let G_2 be another graph with $V(G_2) = V(G_1)$ and $E(G_2) =$

$\{u_{2i}u_{2i+1} | 1 \leq i \leq \frac{n}{2}\}$. Then, it is obvious that $\chi_i(G_1) = \chi_i(G_2) = 2$ and $G_1 \cup G_2 = C_n$ where n is not divisible by 3. Therefore, $\chi_i(C_n) = 4 = \chi_i(G_1) + \chi_i(G_2)$. $\square$

Next, we prove the theorem about the Cartesian product of graphs. The following definition should be given in prior.

Theorem 4.2 *For all graphs G_1 and G_2, we have $\chi_i(G_1 \Box G_2) \leq \chi_i(G_1) + \chi_i(G_2)$.*

Proof: Let $|V(G_1)| = m$ and $|V(G_2)| = n$. $G_1 \Box G_2$ is a graph with mn vertices and two types of edges: from conditions (1) and (2) respectively. The edges of type (1) form a graph consisting of n disjoint copies of G_1, hence its incidence chromatic number equal to $\chi_i(G_1)$. Likewise, the edges of type (2) form a graph with incidence chromatic number $\chi_i(G_2)$. Consequently, the graph $G_1 \Box G_2$ is equal to the union of the graphs from (1) and (2). By Theorem 4.1, we have $\chi_i(G_1 \Box G_2) \leq \chi_i(G_1) + \chi_i(G_2)$. $\square$

We demonstrate the upper bound given in Theorem 4.2 is sharp by the following example.

Example 4.2 Let $G_1 = G_2 = C_3$, then $G_1 \Box G_2$ is a 4-regular graph. If it is 5-incidence colorable, then its square has chromatic number equal to 5 [14]. However, all vertices in $G_1 \Box G_2$ is of distance at most 2. Therefore, $G_1 \Box G_2$ is not 5-incidence colorable and the bound derived in Theorem 4.2 is attained. $\square$

Finally, we consider the incidence chromatic number of the join of graphs.

Theorem 4.3 *For all graphs G_1 and G_2 with $|V(G_1)| = m$, $|V(G_2)| = n$ and $m \geq n \geq 2$. We have $\chi_i(G_1 \vee G_2) \leq \min\{m+n, \max\{\chi_i(G_1), \chi_i(G_2)\} + m + 2\}$.*

Proof: On the one hand, we have $\chi_i(G_1 \vee G_2) \leq m+n$. On the other hand, the disjoint graphs G_1 and G_2 can be colored by $\max\{\chi_i(G_1), \chi_i(G_2)\}$ colors, and all other arcs in between can be colored by $m+2$ new colors. Therefore, $\max\{\chi_i(G_1), \chi_i(G_2)\} + m + 2$ is also an upper bound for $\chi_i(G_1 \vee G_2)$. $\square$

Similar to the previous practices , we utilize the following example to show that the upper bound in Theorem 4.3 is sharp.

Example 4.3 Let $G_1 = K_m$ and $G_2 = K_n$. Then the upper bound $m + n$ is obtained since $G_1 \vee G_2 \cong K_{m+n}$. On the other hand, let G_1 be the null graph of order m and G_2 be the null graph of order n. Then the other upper bound $\max\{\chi_i(G_1), \chi_i(G_2)\} + m + 2$ is attained because $G_1 \vee G_2 \cong K_{m,n}$. $\square$

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